Using Cabri3D Diagrams For Teaching Geometry
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Cabri3D is a potentially very useful software for learning and teaching 3D geometry. The dynamic nature of the digital diagrams produced with it provides a useful aid for helping students to better develop concept images of geometric concepts. However, since any Cabri3D diagram represents three-dimensional objects on the two dimensional screen of a computer, some care is needed in order to avoid serious misconceptions which can arise from its use, in particular those due to the fact that projections do not preserve, in general, angles and distances. In this paper, after comparing digital diagrams (i.e. diagrams on a computer screen) with the more usual diagrams and models, we illustrate an experience around the use of Cabri3D with prospective high school teachers, aimed at clarifying which misconceptions may arise while interpreting a Cabri3D diagram.

1 INTRODUCTION

Three-dimensional Euclidean geometry is not a popular subject nowadays. One of the main reasons for this is that diagrams representing three-dimensional geometric objects are difficult to interpret.

“The survey by the French Ministry of Education shows that the fifteen-year-old students’ most repulsive subjects in mathematics were spatial geometry and statistics. Only ten percent of teachers taught spatial geometry. They said that they did not have enough time to teach it, but the real reason is that the students cannot see in 3D. We mean this, as the students cannot picture spatial situation of a teacher blackboard figure” (Bako, 2003, p. 1).

Insufficient attention is paid to the visualisation capabilities of the students. “It seems that there is a hidden naïve assumption that somehow students do have visual thinking abilities and that they apply visual reasoning when they have to” (Hershkowitz, Parzysz, and Van Dormolen, 1996, p 166).

This assumption is not only naïve but also dangerous, especially in solid geometry, where it is quite easy to pick vague and distorted ideas about geometric objects and relations and it is quite hard to get rid of them.

For helping the development of good concept images of three-dimensional geometric objects, educators have some possible aids: models, manipulatives and diagrams. The recent availability of 3D dynamic software, like Cabri3D (Bainville and Laborde, 2004) gives a potentially important new tool for developing visual education for solid geometry. In this paper we consider some teaching possibilities with respect to Cabri3D and compare them with the use of models and diagrams. We shall particularly be interested in misconceptions which may arise from interpreting Cabri3D diagrams. In section 5 we give a precise statement of the claims we tested.

Glossary of terms

For the sake of clarity, we briefly recall the meaning we assign to some crucial terms.

• Models: concrete objects representing concrete instances of geometric objects like wood polyhedra, plastic triangles and so on.

• Manipulatives: set of concrete objects over which one can perform manual activity in order to represent geometric objects, constructions and relations. For example: Geoboard, Mira, Polymorf, Polydron, the look-through-window.

• Diagrams: drawings representing geometric objects and relations on a sheet of paper. Diagrams may be further subdivided into:

  • Sketch diagrams, i.e. outline drawings representing some geometric features of two or three-dimensional geometric objects, possibly with the aid of graphic conventions.

  • Euclidean 2D diagrams, i.e. diagrams built by means of ruler and compass, which faithfully represent one particular instance of some geometric 2D configuration;

  • Euclidean 3D diagrams i.e. diagrams built by means of ruler and compass, which represent one particular instance of some geometric 3D configuration by means of the methods of descriptive geometry. These in general are not a faithful representation although some methods of projection can faithfully preserve some of the properties of the original 3D configuration, like parallelism.
• Digital diagrams: diagrams representing geometric objects and relationships on the screen of a computer by using some software. We further subdivide digital diagrams according to the software, or group of software, used to produce them, even if a more conceptual subdivision is possible. We shall speak for instance of Cabri3D diagrams, CAD diagrams, MSPaint diagrams and so on. For simplicity we shall denote the software Cabri Géomètre (Laborde and Bellemain, 2001) by Cabri 2D.

• Concept Image: “The total cognitive structure that is associated with the concept which includes all mental pictures and associated properties and processes. It is built over the years through experience of all kinds, changing as an individual meets new stimuli and matures” (Tall, 1991, p.7). In geometry, the ideas of concept image and concept definition introduced by Tall and Vinner (1981) have been included in the general cognitive theory of Figural Concepts developed by Fischbein. “In geometry the ideal figural concept corresponds with the concept definition, while its mental reflection, with all its connotations and ambiguities corresponds to a concept image” (Fischbein, 1993, p. 150).

• Visual Education: education of the ability to visualise real objects, mathematical concepts, processes and phenomena (see Hershko 2001a; Mariotti witz, Parzysz, and Van Dormolen, 1996). In this paper we are interested in visual education under the perspective of interaction with real shapes in space, i.e. for helping the formation of good and harmonic concept images of geometric objects. This is just a first step in the direction of a complete mastering of concept images, which can be seen as the ultimate goal of geometry. However, this step is an important and often neglected one, especially in 3D geometry. Manipulatives, diagrams and digital diagrams may contribute positively to the shaping of good concept images of geometric objects and to the exploration of their relationships. We shall discuss some of their merits and limits in the following sections.

2 MODELS AND MANIPULATIVES

The use of models and manipulatives is an essential aid in learning geometry, according to all major theoretical perspectives. There is also wide support that it facilitates the construction of sound representations of geometric concepts in young children (Grenier and Denis, 1986; Gerhardt, 1973; Prigge, 1978). There is empirical support that manipulatives are an essential aid in learning geometry also for older students, especially those at lower levels in the van Hiele hierarchy (Clements and Battista, 1992; Fuys, Geddes and Tischler, 1988). Models and manipulatives are especially useful for teaching solid geometry (Parzysz, 1988; 1991).

“Use of manipulatives seemed to allow students to try out their ideas, examine and reflect on them, and modify them. This physical approach seemed to maintain students’ interest, to assist students in creating definitions and new conjectures, and to aid them in gaining insight into new relationships”, (Clements and Battista, 1992, p.449).

However the usefulness of models and manipulatives for 3D geometry is severely limited by their rigidity. For example, looking inside a 3D model or cutting it along precise planes is something which one would like to be able to do at an early stage but cannot usually be done.

3 DIAGRAMS

Diagrams, especially in the teaching of plane Euclidean geometry, may give students an immediate, intuitive grasp of many geometric ideas. However, diagrams, at the first stages of learning, are less effective than manipulatives. One reason may lie in their “nonmanipulability” (Clements and Battista, 1992, p.448). For example, students tend to think only of squares with sides which are parallel to the margin of the sheet on which they are drawing. This rigidity of diagrams when visually interpreting a theorem, for example about squares, may result into an inability to recognise that the theorem may still be true when the sides of the square are not parallel to the edges of the sheet of paper. Situations of this kind are well known, and are described for example in Clements and Battista (1992, p.448).

Moreover, diagrams need to be suitably interpreted; inessential information must be discarded in order for diagrams to be useful for problem solving (see Kabanova-Meller, 1970; Vladimiriskii, 1971).

“Diagrams can affect students’ representations of concepts, theorems and problems … [and] misconceptions could arise out of students’ interpretations of the diagrams themselves” (Clements and Battista, 1992, p. 448).

This is particularly true for diagrams representing 3D objects (Parzysz, 1988). We mention three reasons for which diagrams representing 3D objects are difficult to interpret correctly.

• Loss of information due to projection. What you see in a diagram is a projection of what you have in space and under this projection each point of the plane corresponds to infinitely many points of the space.

• Non displayed parts of an object. For example, since one needs to use limited objects in a diagram, planes are usually represented as quadrilaterals. This has the consequence that many students find natural the idea that two planes may intersect at a segment or that four points are needed to determine a plane.

• Nonmanipulability. As we said above, this issue is common to diagrams for 2D objects.

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Even diagrams for simple 3D configurations may be very difficult to interpret. We cite an example reported in Bako, 2003. Figure 1 was given in a survey to teachers at the IREM of Strasbourg; it was explicitly said that it represented a cube and some easy questions about the figure were raised. Most of the students’ answers implied that they thought that points G, N, M and P were collinear.

Figure 1  The diagram used in the survey conducted by the IREM of Strasbourg. Are points G, N, M and P collinear?

The problem of representing space on the plane has been considered over the centuries by mathematicians, painters, architects and engineers and eventually evolved in the theory of perspective during the Renaissance. It was deeply studied also from a purely mathematical point of view, giving rise to the subject of descriptive geometry. Descriptive geometry was a basic subject for mathematicians, engineers and architects for years (in Italy it was part of the standard curriculum for mathematics teachers until 1961). In particular it gave high school mathematics teachers the basic intuitive background for 3D geometry and the basic skills for drawing meaningful diagrams of 3D objects. As a consequence of the withdrawal of descriptive geometry from the university curriculum, teachers’ ability to use meaningful drawings for illustrating 3D objects has rapidly deteriorated, resulting in a progressive restriction of the intuitive background of the students and a consequent higher demand on their abstraction abilities. Teachers began to assume that students were able to imagine objects and constructions only by reasoning since suitable models were often not available, and representing and understanding 3D objects with diagrams has become very difficult for both. At the same time, however, the teaching of plane geometry became more and more deficient in providing the abstraction required for grasping even a basic development of rational solid geometry. These are some of the reasons why the teaching of 3D geometry is nowadays much neglected.

After years of oblivion, the exclusion of the methods and spirit of Descriptive Geometry and its role in the visual education of students has led some mathematics educators to reconsider the use of diagrams in combination with the use of models and manipulatives for teaching 3D geometry (see Bako, 2003; Bartolini Bussi and Maschietto, 2006; Hershkowitz, Parzysz and Van Dormolen, 1996; Parzysz, 1988; 1991).

These studies show that students need to develop good conceptual images of three-dimensional geometric objects through constant use of 3D models and manipulatives, until they become able to imagine the spatial situations without having the models and manipulatives before their eyes. Moreover, in order to profitably use diagrams, they need an explicit training in the projective technique for planar representation of space. This training needs to be supported by the use of material models and instruments, like cords, poles, rulers, and the look-through-window. This traces a long path towards the attainment of a reasonable degree of competence in space geometry.

We are interested in exploring a more modest alternative path for providing a quicker access to some aspects of 3D geometry by making use of computer graphics technology and in particular Cabri3D.

4  DIGITAL DIAGRAMS

As we said in the previous section, non digital diagrams of 3D objects suffer some severe limitations, among which are: non-manipulability (the same problem exists for diagrams of 2D objects), loss of information due to projection and non displayed parts of an object. In this section we shall see if digital diagrams are better than ordinary diagrams and under which conditions.

Even if in Cabri 2D and Cabri3D diagrams there is much more than the figural component, our main concern in this paper will be on the aspects associated with visual education; for more on Cabri 2D diagrams see Arzarello, Olivero, Paola and Robutti, 2002; Laborde, 1992; Laborde, 1993; Laborde and Laborde, 1991; Laborde and Strasser 2001a; Mariotti, 1990; Mariotti, 2001b and Mariotti, 2005. For example, in Mariotti, 2005 one can find a detailed analysis of the positive role that Cabri constructions can play within the dialectics between figural and conceptual in geometry.

In order to overcome the non-manipulability of the usual (i.e. non digital) diagrams, educators pointed out that diagrams need to be varied, so that students are not led to form incorrect concepts. Computers can be of great help since they can produce manipulable digital diagrams (see Clements and Battista, 1993, p. 448).

Much geometrical software exists. In this paper we consider only Cabri 2D and Cabri3D.

Cabri 2D diagrams. A small revolution in the teaching of plane geometry took place when the first dynamic geometry software appeared. With Cabri 2D, perhaps the best known of this genre, one can draw geometric objects, like points, lines, and circles and perform basic constructions, like joining two given points with a line; finding the intersection point of two lines; drawing a circle of given centre and radius. A Cabri 2D diagram is produced by performing a set of constructions over a set of basic objects, and the way to produce it closely resembles the...
classical constructions with ruler and compass. A Cabri 2D diagram however is much more than a diagram produced by ruler and compass. The diagram is in fact manipulable. It can be changed by dragging elements around the screen. Dragging preserves the relations between the elements, hence Cabri 2D provides a valuable environment for making conjectures, finding counterexamples, introducing proofs, and so on.

More precisely, the construction of a Cabri 2D diagram requires that the geometric concepts needed for the construction are made explicit and turned into a sequence of explicit instructions for the software. These same constructions determine both the procedure for representing an instance of a geometric configuration and define the logic of the configuration itself. Since the logic of the configuration is preserved under dragging, the visual appearance of the configuration may change but the logic behind it, i.e. all the geometric relationships, are preserved. Under the perspective of the theory of figural concepts (Fischbein, 1993), one can say that both the figural and the conceptual component are contained and evoked in any Cabri 2D diagram (see Mariotti, 2005).

**Cabri3D diagrams.** The dynamic software Cabri3D is an attempt to build a microworld for three-dimensional geometry based on general principles which are similar to those implemented in Cabri 2D. There is a set of primitive objects (points, lines, planes, spheres) and a set of elementary operations (joining two given points with a line, intersecting two planes, finding the line through a point and perpendicular to a plane, and so on). With Cabri3D one builds a three-dimensional geometric configuration by repeatedly applying the elementary operations to some given primitive objects. These configurations are displayed on the computer screen by digital diagrams and the drawing of these diagrams is automatically performed by the software according to the rules of descriptive geometry. We consider now some crucial issues about Cabri3D diagrams.

**Placing points in space, interacting with a Cabri3D diagram.** Any projection map from the three-dimensional space to the computer screen is not invertible. Therefore in order to place a point into a 3D scene one needs more than just clicking on the screen. Cabri3D allows only placing points on a 1 or 2 dimensional object. First one selects the object on which one wants to position the point. When the object is selected, the projection map becomes locally invertible; hence a point can be placed on the 3D configuration by just clicking a point on the screen. If one wants to place a point but does not want to bound it to a one or two dimensional object, there is a complicated procedure which will be explained in Section 8.

**Dragging and Cabri3D manipulable diagrams.** Dragging is implemented as in Cabri 2D. A Cabri3D diagram is manipulated by dragging a point. The point moves, the diagram changes but all geometric relationships are preserved.

**Cabri3D diagrams and the loss of information due to projection.** In order to produce accurate diagrams with Cabri3D, the user does not need any knowledge of descriptive geometry. However the user still has the problem of interpreting the diagrams. This is greatly facilitated by the ability to change the point of view from which the 3D scene is drawn. Points of different objects or different points of the same object can be projected onto the same point of the screen. The ability to change the point of view, i.e. to “look around an object”, can be used to separate points that were previously identified. This implies that many properties of a Cabri3D diagram can be correctly established without possessing any notion of descriptive geometry. For example, by changing the point of view, anyone can easily interpret Figure 1 correctly, as Figure 2 shows.

In the first diagram the segments GN, NM and MP are projected onto the same line. By changing point of view it becomes clear in the second diagram that the three segments are not collinear in space. Note moreover that the solid appearance of points and segments in each diagram and its rendering helps the correct interpretation of the first diagram itself.

However, as we shall see in the description of our experiments, changing the point of view is not sufficient to see all the properties of a configuration in a Cabri3D diagram and, moreover, without any knowledge of descriptive geometry, some serious and unexpected misconceptions may arise from Cabri3D diagrams.

We shall consider four kinds of properties:

- **Combinatorial properties:** properties related to the number of objects in a configuration. For example, the number of vertices, sides and faces in a polyhedron.

- **Topological properties:** properties invariant under continuous deformation. A non combinatorial topological property is, for example, the dimension of an object.

- **Affine properties:** properties invariant under the affine group. For example, parallelism of lines.

- **Euclidean properties:** properties invariant under the Euclidean group. For example angles between lines and distance between points.
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Projections do not in general preserve any of these properties but the possibilities of changing the point of view, changing the projection method, dragging and making conceptual constructions combined with lightening and rendering 3D objects on a computer screen can help to see, or at least guess, some of the properties of a configuration from its Cabri3D diagram, and to resolve some of the misconceptions which may arise from the usual diagrams.

Cabri3D diagrams and non displayed parts of an object. In Cabri3D planes are represented, as usual, by quadrilaterals. In Figure 3 we see the quadrilateral representing the base plane.

![Figure 3 The base plane z = 0 is represented by a quadrilateral.](image)

However, the existence of non displayed parts of some objects causes the impression that intersections disappear each time they occur outside the visible part of the object, as in Figure 4.

![Figure 4 An intersection outside the visible object](image)

In Figure 4, the line \( r \) through point \( P \) is parallel to the vector \( V \) in the base plane \( a \). Point \( Q \) belongs to \( r \) and we constructed the line through \( Q \) perpendicular to the base plane \( a \). Since the intersection of this line with the plane \( a \) is outside the displayed part of \( a \), it seems that the line and the plane do not intersect.

5 WHAT WE TESTED AND EXPLORED IN THE EXPERIMENTS

The experiments we describe in this paper were conceived in order to explore which properties of a 3D geometric configuration can be seen in Cabri3D diagrams and which misconceptions may arise. We tested the following hypothesis:

Without any previous knowledge of descriptive geometry, and by only employing the default method of projection, the only difficulties arising in interpreting Cabri3D diagrams are associated with the Euclidean properties of the diagrams, (i.e. distances and angles). Hence, combinatorial, topological and affine properties are correctly detected.

As we shall see our conjecture proved to be too optimistic. The experiments were also aimed to:

- understand if the dragging facilities could also help students detect some Euclidean properties of the original configuration in a Cabri3D diagram;
- understand if Cabri3D helps students resolve previous misconceptions, like those generated by representing planes with quadrilaterals; understand what kind of misconceptions may be generated by Cabri3D diagrams; understand which properties are more likely to generate these misconceptions;
- investigate if students feel the need for models while interpreting Cabri3D diagrams;
- observe if students are aware of the problems associated with placing points in space when interacting with Cabri3D diagrams.

The reason why we are interested in this kind of investigation is that, as we said in the introduction, we believe that students need to develop visual thinking abilities about the properties and relationships of spatial configurations. The lack of such visual abilities makes the teaching of 3D geometry ineffective. Since the traditional methods of developing these abilities are inadequate or need a long training to be effective, we want to understand the role that Cabri3D can play in developing the visual thinking abilities of students.

6 THE EXPERIMENTS

Subjects Our experiments were conducted with eight prospective high school teachers who had a mathematics degree and were attending the second year of a two-year postgraduate teacher training course (SSIS). All of them had studied at least a semester course in linear algebra and one in 2D and 3D analytic geometry.

All students were already acquainted with Cabri 2D from the previous year. Their activities with Cabri 2D are analysed in Accascina, Margiotta and Rogora, 2005.

Before conducting the experiments with Cabri3D we tested the students with questions about 3D geometry taken from multiple choice tests used for helping high school students in the choice of university studies (see Accascina, Mastrogiovanni and Rogora, 2004; Accascina, Rogora et al., 2004). Moreover we asked them to write the definition of perpendicularity of two lines, of two planes and of a line and a plane.
We discussed the results of the test with the students themselves and it turned out that, as we expected, students’ ideas about the notion of perpendicularity of planes and lines were quite confused, their ability to imagine 3D objects was weak and their awareness of the reasons behind the definitions of 3D geometry was low. We decided to enter into some discussions in order to clarify these points. We discussed, among other things, how to define perpendicularity between linear subspaces and proved some basic facts before beginning the experiments with Cabri3D.

**Organisation.** We ran the experiment in two distinct meetings of three hours each in a computer room. Each student had access to a PC and we used a projector linked to the instructor’s PC. In the first meeting each student worked on their own. In the second meeting we grouped the students into two groups. The proposed activities were described in labsheets, handed to the students (which are available on request). Each student received a plastic framework of a cube. Opaque and transparent solid models were at students’ disposal. We were interested to see if and for which questions students would have preferred to use models instead of Cabri3D.

**Data collection.** Students were asked to provide written answers to all questions on the labsheets. We planned time for discussion at the end of the first meeting and during the second meeting. Both authors took notes of the discussion during the experiments.

**Activities.** In all activities we chose to ask for constructions which could be performed without the need to discuss the problem of moving “points in space” (see Section 8). We only made students aware of the problem in the last activity.

**Introduction to Cabri3D.** Students quickly learned the basic usage of the program and were able to easily perform the simple tasks required in their first labsheet.

**The heights of tetrahedron and a decomposition of the cube.** We showed the students a Cabri3D diagram of a regular tetrahedron and its heights. Then we moved the tetrahedron to a non regular one. The experience is described in Accascina, Margiotta and Rogora, 2005. Then we showed a Cabri3D diagram of a square based pyramid inside a cube and asked them to use Cabri3D in order to decompose the cube into three pyramids equivalent to the given one, see Figure 5.

In both these activities Cabri3D brilliantly outperformed the use of drawings and models. Students were highly satisfied with the insight they got.

**Plane sections of the cube.** We planned this activity for testing our main hypothesis. The results of this activity will be discussed thoroughly in the next section. In this activity we did not ask for any construction but only to interact with a Cabri3D diagram by changing the view point and dragging.

**Placing points in space.** We made students aware of the problem and discovered an unexpected difficulty in interpreting Cabri3D diagrams.

![Figure 5](image1.png) In the first diagram the cube with a pyramid. In the second diagram, a second congruent pyramid constructed by the students.

**7 EXPERIMENT ON THE PLANE SECTIONS OF THE CUBE**

The study of properties of plane sections of a cube is a classical elementary problem in 3D geometry, which has received attention both from teachers, as an intriguing elementary example, and from researchers in mathematics education (see Bako, 2003; Castelnuovo, 1985; Ehrenfeucht, 1964). From our point of view, this problem is interesting because it raises a range of questions of different nature: combinatorial, affine and Euclidean.

We prepared a Cabri3D diagram in order to allow students explore plane sections of a cube by dragging and changing the point of view, see Figure 6.

![Figure 6](image2.png) The plane section is controlled using the point $V$ on the sphere and the point $P$ on the line $r$.
intersection of this plane with the cube is displayed. The interaction with the construction in the diagram is through three controls. The first, provided by Cabri3D, allows one to change the point of view on the diagram, i.e. to move around it. The second depends on the position of the point \( V \) and controls the orthogonal direction to the section plane. The third depends on the position of the point \( P \) and controls the position of the plane in the pencil of planes orthogonal to \( r \). By dragging \( V \) and \( P \) (and possibly changing the point of view) the plane of the construction can be any plane. The interaction with the diagram was considered very intuitive by the students. It took them some practice, however, to be able to position the planes in a satisfactory way (for example it is not easy to place a plane exactly through the mid-points of three sides). However we expected these difficulties and were interested to observe how the students were able to cope with them.

**First Meeting.** During the first meeting we asked the following questions.

- What kind of remarks can you make about the polygons which can be obtained as plane sections of a cube? **Goals:** Which properties can easily be detected with Cabri3D? Do students prefer to use Cabri3D or a model to answer this question? **Analysis:** All students decided to work only with Cabri3D and not with the plastic model of the cube we provided them. All of them found polygons of 3, 4, 5 and 6 sides. Most of the remarks were about different types of quadrilaterals: they saw squares, rectangles, trapezoids but only one of them claimed to see parallelograms. The only remark about triangles was about the existence of equilateral triangular sections. No remark was made about pentagons except raising the question of the existence of regular pentagonal sections (one student). Two students made the remark that regular hexagonal sections exist. No one claimed to have seen a right triangle, which cannot occur as a plane section of the cube but it is often claimed to be seen when one of the vertices of a triangular section (like the one in Figure 6) approaches the vertex of the cube (Bako, 2003). Two of the students made some remarks about the topological properties of the function \( n(V, P) \) which counts the number of sides of the section with the plane corresponding to the position \( P \) and \( V \) of the controls. They asked themselves how this number varies when the sections are obtained by fixing \( V \) and moving \( P \), i.e. in a pencil of parallel sections, and what kind of symmetries it has.

- Can a heptagon be the plane section of a cube? Can a plane section have more than seven sides? **Goals:** Which argument, if any, is given to answer this question? Do students need to use Cabri3D? **Analysis:** All students said that no polygons with more than 6 sides can be cut by a plane on a cube. Four students gave a proof of this (the number of sides is linked to the number of faces of the cube) and most students said that Cabri3D was helpful.

- Can you obtain a square? And a rectangle which is not a square? **Goals:** Is Cabri3D useful to answer this question or do students need a model? **Analysis:** All students saw square sections. All but one saw rectangular sections. Two students did not use Cabri3D. Those who used Cabri3D correctly detected right angles in the plane section. No one used the plastic model.

- Can you obtain a rectangle whose side ratio is \( \sqrt{2} \)? **Goals:** Is Cabri3D useful for answering this affine question? **Analysis:** All students worked only with Cabri3D and found trapezoids.

- Can you obtain a quadrilateral which is not a parallelogram? **Goals:** Is Cabri3D useful to answer this question? Do misconceptions about lines in perpendicular planes persist when using Cabri3D or can it help to get rid of them? **Analysis:** This was the crucial question. The existence of parallelograms which are not rectangles cannot easily be detected with Cabri3D. We were interested to see if students would have used Cabri3D or if they would have begun to think without using it. Only one of the students claimed the existence of a section which is a parallelogram but not a rectangle. She was the only one who found Cabri3D useful to answer this question, but unfortunately she did not explain why.

**Second Meeting.** During the second meeting we decided to divide the students into two groups. The reason for this was that we wanted to follow the discussion between the students in each group about the crucial question if there are sections of the cube which are parallelograms and not rectangles. We wanted to see if and how students would have used Cabri3D and models to support their arguments in the discussion. We formed the groups by taking into account the results of the multiple-choice test: we put the best two in different groups, the worst two in different groups, and so on. The students of the first group began to use Cabri3D very soon, trying to convince themselves that all the parallelogram sections were rectangles. They were not able to
disprove this statement by using Cabri3D. In fact they were not able to decide if the angles they saw on the screen were not right angles because the angles in the 3D scene were not right angles or because of the changes due to the projection onto the computer screen. When they tried to convince themselves that the sections were actually rectangles they made a basic mistake by assuming that every plane intersects two perpendicular planes in two perpendicular lines. This was quite amazing since only a couple of weeks before we had discussed at length orthogonal planes and lines in orthogonal planes. It seemed that Cabri3D was not effective in helping them recover from their basic misunderstanding. Moreover, when we explicitly reminded them that the plane sections of two perpendicular planes are not necessarily perpendicular, one student insisted on her mistake, continuing to look at the diagrams produced by Cabri3D. We were able to convince her only when we asked her to compute the sum of the internal angles of a triangular section of the cube in the hypothesis that every plane cuts two perpendicular planes in a right angle. None of them felt the necessity to select a physical model. This suggests that Cabri3D should not be used in a classroom in the same way as suggested by Sinclair for a 2D dynamic-geometry software:

“The dynamic-geometry supported classroom offers a challenge regarding the creation of the contexts mentioned by Papert. Students in such classes may spend much of their class time interacting with a computer program rather than communicating with a teacher” (Sinclair, 2003, p. 289).

We believe that students using Cabri3D should communicate frequently with their teachers in order to avoid the manifestation of basic misconceptions.

In the second group the situation was different. They did not make the mistake about sections of orthogonal planes and did not use Cabri3D for about 20 minutes during which time they discussed the problem until they got a clear idea of what they wanted to see, i.e. the most distorted (non rectangular) parallelogram they could conceive as a plane section of the cube. One of them understood how to find such plane sections by looking at a plastic model of the cube and then tried to reproduce it with Cabri3D. She was not convinced that the parallelogram was not a rectangle by looking at the model but she believed that reproducing the section with Cabri3D would have convinced her. Once she got the section with Cabri3D, she moved the point of view perpendicularly to the cutting plane (i.e. until the line r perpendicular to the plane appeared in the diagram as a point) and then she convinced herself and the other members of her group that the section was not a rectangle. The section, shown in Figure 7, was obtained by taking two opposite vertices A and B of the cube, the midpoint C of any of the 6 edges adjacent to neither A nor B and taking the section with the plane ABC. Then they gave also a formal proof of this by noticing, as it became evident from the diagram and by using Pythagoras’ theorem, that the lengths of the two diagonals of the parallelogram section are $\sqrt{2}$ and $\sqrt{5}$ respectively.

Figure 7 A parallelogram which is not a rectangle. The point of view is chosen in such a way that the line r is projected onto the point O. This convinced the students that non rectangular parallel sections exist.

It seems therefore that Cabri3D may also help the investigation of some Euclidean properties of 3D objects but one needs to be able to dispel basic 3D misconceptions before using it effectively.

8 EXPERIMENT ON THE PROBLEM OF PLACING POINTS IN CABRI3D (v. 1.0.3)

We obtained further evidence of the fact that basic geometric properties can be difficult to see with Cabri3D in our last labsheet which was produced in order to study the problem of placing points in space through a Cabri3D diagram.

We conducted the following experiment. We asked students to open a blank file (not the default file but one with no elements), draw a tetrahedron and describe its properties. They made the following construction (see the first diagram in Figure 8). They drew four points and then connected all pairs with segments. We did not explain that each point belongs by default to the base plane ($z = 0$). The students were well aware that in order to investigate the properties of a diagram they had to change the point of view and that changing the point of view does not change the objects. By moving the point of view, each of them saw images like those in the diagrams of Figure 8 but none of them was able to conclude that the tetrahedron was degenerate (i.e. contained in a plane) even if this became absolutely evident when we pointed it out to them.
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It is hard to believe that even by looking at images like those of the panels of Figure 8 they did not notice that the tetrahedron was contained in a plane, but this is what happened.

After telling them that the four points were on a plane we asked them to try to explain why the software behaved in such way. They seemed not to be aware of the fact that the interaction with a 3D diagram through the two dimensional screen of a computer cannot be made by just clicking on the mouse because the information provided by a point on the screen (two Cartesian coordinates) is not in itself enough to identify a point in space (three Cartesian coordinates). We had to explain in detail that this is not due to bad software design but to the unavoidable fact that the map from space to the plane is not invertible.

Figure 9 When one drags a “point in space”, the Cartesian coordinates of the point appear. When the point is dragged with the shift key pressed only the last coordinate changes. When the point is dragged with the shift key released, only the first two coordinates change. By combining the two movements all positions in the space can be reached.

10 CONCLUSIONS

Our students liked to use Cabri3D very much and found it user friendly. Some of them said they had finally found something useful for exploring the geometry of space, since they always found it very hard to study 3D geometry with only the aid of static diagrams.

However we have noticed that Cabri3D may be insidious since one can see things which are not true (perpendicularity of plane sections) and cannot see quite evident facts (degeneracy of tetrahedra).

In our experiment on plane sections of the cube we have seen that Cabri3D is useful for understanding combinatorial, topological and affine properties of a 3D construction. However the experiment on the degenerate tetrahedron showed that even simple non metric properties like belonging or not to a plane may not be seen.

The search for Euclidean properties in a diagram can become more misleading than the search for combinatorial, topological and affine properties. We noticed, for example, that misconceptions about lines in perpendicular planes are not always resolved by Cabri3D, and worse, that some students appeared to have made a step back in their understanding when using Cabri3D, as shown in the second meeting by the students of the first group.

These first experiments led us to think that we were too optimistic in our basic conjecture. We observed that there are also some non Euclidean properties which cannot be detected in Cabri3D diagrams, as in the experiment on degenerate tetrahedra. Further experiments on this are needed.

We now make some remarks on the other issues we raised in Section 5.

• About Euclidean properties and dragging facilities. We think that Cabri3D may sometimes be useful for understanding some Euclidean properties of a 3D construction, since some Euclidean properties may be seen
dynamically. For example, the existence of rectangles whose side ratio is 2 was seen by some of our students by looking at the dynamic change of a rectangular section whose side ratio was $\sqrt{2}$ into a degenerate one whose side ratio was infinite. Other Euclidean properties may be seen, or at least guessed, by suitably changing the point of view, as shown in the second meeting by the students of the second group (Figure 7).

- **About misconceptions.** We have already noticed that incorrect ideas about lines in perpendicular planes are not always clarified when using Cabri3D. We also investigated the growth of misconceptions related to the representation of planes with quadrilaterals. During our experiments we heard comments like “this line, perpendicular to this plane, does not intersect the plane”, when students looked at diagrams like that of Figure 4. These moments of confusion are of course easily overcome by people with sufficient mathematical knowledge, but they are possible sources of dangerous misconceptions for less prepared students. However Cabri3D itself can help with problems like this. For example, referring back to Figure 4, we can ask the software to compute the intersection of the line and the plane. The point of intersection appears also when it is outside the displayed part of the plane as in Figure 10.

![Figure 10](image)

**Figure 10** If we ask the software to compute the intersection, the point $X$ appears also when it is outside the displayed part of the plane $a$.

Also misconceptions like “A plane is determined by four points”, which we sometimes hear in our university examinations, can possibly be conveyed through the representations of planes through quadrilaterals but we have not yet explored this aspect in our experiments.

- **About models.** Cabri3D clarifies issues which are not easily understood with models and diagrams, like the problem of decomposing a cube in three square based pyramids or the nature of plane sections of a cube. However we have also noticed that Cabri3D may weaken the need to refer to concrete models to understand situations which cannot be clarified by looking at Cabri3D diagrams alone, like the existence or not of parallelogram sections which are not rectangles. This is what happened in the second meeting with the students of the first group. They did not feel any need to use 3D models instead of Cabri3D. Even when they got confused, as when looking for parallelograms which were not rectangles, they did not think they would gain a better understanding with a 3D model. As we said, the behaviour of the second group in the same experiment was different, but since among the students of the second group there was one who turned out to be much more mathematically gifted than the others, we believe that the behaviour of the first group can be assumed to be more typical. Some experiments with high school students seem to confirm this.

- **About placing points in space.** We believe that the problem of placing points in space is a potentially confusing problem. Moreover the students seemed quite reluctant to accept the rationale behind it. We suggest to avoid the problem at the beginning and to introduce it in a gradual way.

We found that Cabri3D was very effective for quickly introducing students to 3D geometry and giving them enough intuitive support for understanding non-trivial mathematical situations which usually are not well grasped. In fact we also repeated the experiment of plane sections of the cube with a class of high school students without any previous knowledge of three-dimensional geometry; they enjoyed the activities and were able to grasp the main intuitive ideas. However the teacher must be conscious about confusions and misconceptions which may arise when interpreting a Cabri3D diagram. This is necessary in order to use the software without generating confusion in the students’ minds.

One final remark. We have noticed that the use in class of Cabri 2D is very different from the use of Cabri3D. Since Cabri 2D represents 2D objects and constructions faithfully students tend to think that one does not need to provide a formal proof of what one sees on the screen. Therefore careful planning is needed by the teachers in order to introduce students to proofs (see for example Mariotti, 2001a; 2001b). On the other hand, since Cabri3D gives a non faithful representation of 3D objects and constructions, confusion and misconceptions easily arise. Therefore, during classroom activities with Cabri3D, students tend to become very demanding with respect to their teachers. During activities with Cabri3D many questions arise for which is not easy to find convincing arguments based on models or Cabri3D diagrams for answering the question in an intuitive way. For example questions about the existence of sections which are parallelograms but not rectangles or the non existence of triangular sections with a right angle. These questions however give the opportunity to make students aware of the necessity of abstract mathematical reasoning for answering a natural question in a satisfactory way.
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REFERENCES


Grenier, D. and Denis, B. (1986) Symétrie orthogonale: des élèves français et japonais face a une même tache de construction, Petit x, 12, 53-56.


Parzysz, B (1991) Espace, géométrie at dessin. Une ingénierie didactique pour l’apprentissage, l’enseignement at l’utilisation de...


**BIOGRAPHICAL NOTES**

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